AC electric circuits

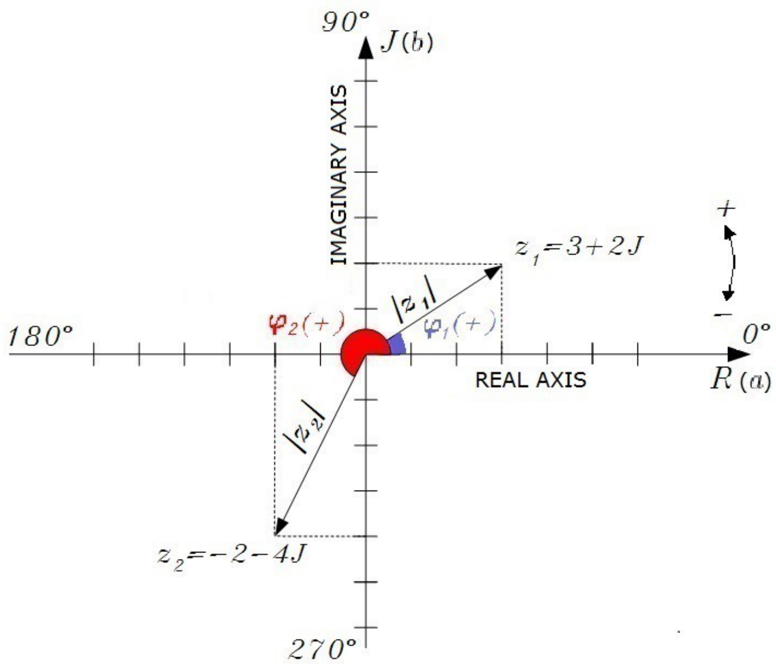
Quick introduction to complex numbers

Alternating current circuits can be solved with the same tricks used for DC circuits, but they are a bit different.

While in DC circuits all quantities are just real numbers, in AC circuits we have to operate with **complex numbers**, also called **vectors**.

Basically a complex number is a binomial expression made of a **real part** *(just a number)* and an **imaginary part,** which is a multiple of the imaginary unit **j,** so called because it is the square root of -1 *(nonsense in real numbers).*

Complex numbers can be represented graphically on the vector compass, which is made of a real horizontal axis and an imaginary vertical axis.



$|z| = \sqrt{a^2 + b^2}$ $\quad z = a + bJ$ $\quad |z| =$ length or modulus

$J = \sqrt{-1}$ $\quad J^2 = -1$ $\quad \varphi =$ angle

$cos\varphi = \frac{a}{|z|}$ $\quad sin\varphi = \frac{b}{|z|}$

They can be converted from the binomial to the polar form *(lenght, angle)* and vice-versa by the Pythagorean Theorem or the trigonometry.

As binomial expressions complex numbers can be added, subtracted, multiplied and divided, but we have to assume that **j^2 = -1.**

In the following division both quantities are multiplied by the **complex conjugate** of the denominator, which becomes a real number:

$$\frac{3+2j}{2-3j} = \frac{(3+2j)(2+3j)}{(2-3j)(2+3j)} = \frac{6+9j+4j+6j^2}{4+6j-6j-9j^2} =$$

$$= \frac{6+13j+6(-1)}{4-9(-1)} = \frac{6+13j-6}{4+9} = \frac{13j}{13} = j$$

Impedance

It is analogue to the resistance in a DC circuit.

The impedence of an AC circuit is represented by a vector *(usually indicated with the letter z)*.

The real part is related to the resistance; it is always positive.

The imaginary part is related to the capacitive or inductive reactance.

An impedance can be pure resistive, pure capacitive, pure inductive or mixed.

Pure resistive

The vector features a positive real part, which is related to the resistance.

$Z_R = R$

Pure capacitive

$$X_C = \frac{1}{2\pi f C} \qquad Z_c = -X_C j$$

The vector is only made of a negative imaginary part and the coefficient corresponds to the reactance of the capacitor.

X_C=capacitive reactance [Ω] C=capacitance [F] f=frequency [Hz]

Pure inductive

The vector is only made of a positive imaginary part and the coefficient corresponds to the reactance of the inductor.

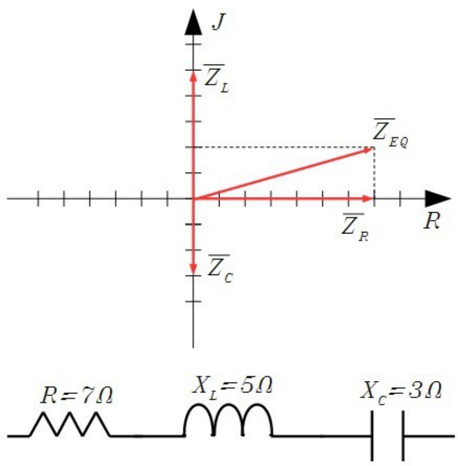
$$X_L = 2\pi f L \qquad Z_L = X_L j$$

$X_L =$ inductive reactance [Ω] $L =$ inductance [H]

Equivalent impedance

Series: $Z_{EQ} = Z_1 + Z_2$ Parallel: $Z_{EQ} = \frac{Z_1 \cdot Z_2}{Z_1 + Z_2}$

The example below shows a RLC series:



$Z_R = 7$

$Z_L = 5J$

$Z_C = -3J$

$Z_{EQ} = Z_R + Z_L + Z_C = 7 + 5J - 3J = 7 + 2J$

$Z_{EQ} =$ series equivalent impedance

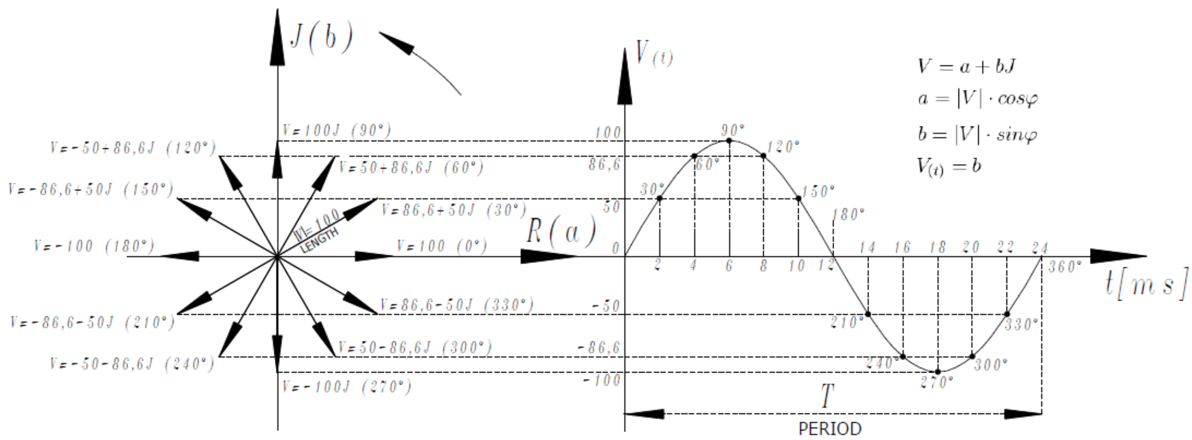
AC voltage and current

Voltage and current are alternating quantities, whose variation is represented by a **sine wave**, related to a **rotating vector**. This one spins with an angular speed associated with the frequency.

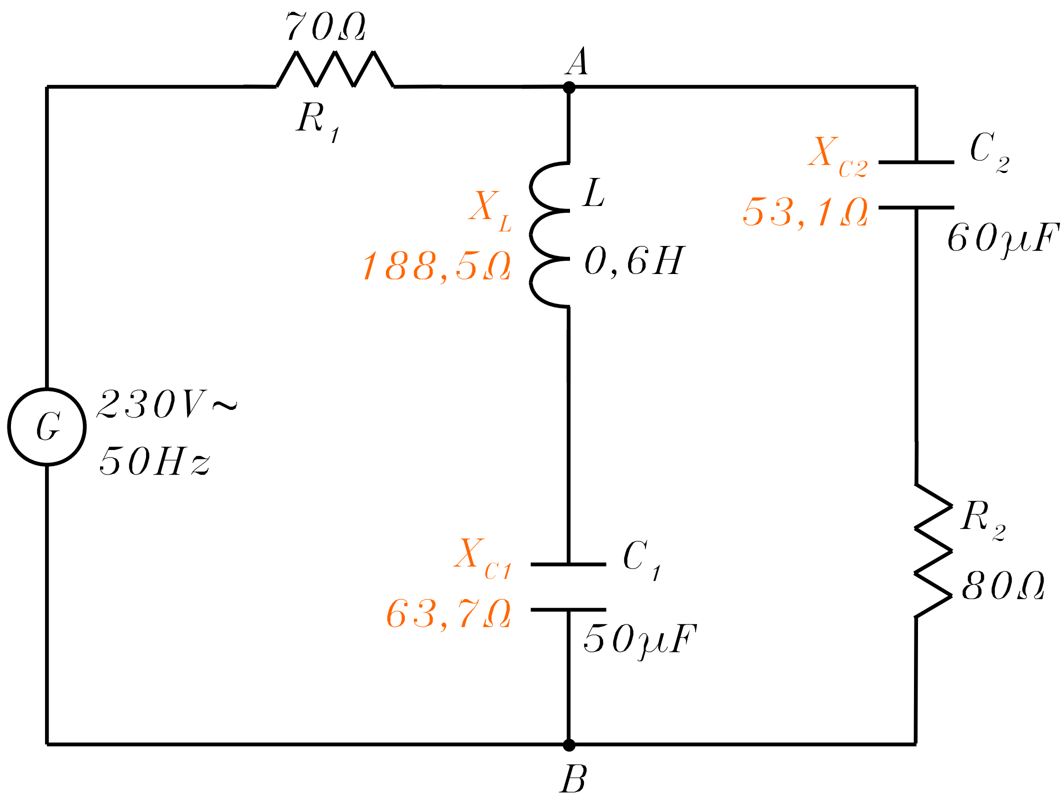
$$\omega = 2\pi f$$
$$\omega = \text{angular speed [radians/second]}$$
$$2\pi \text{ radians} = 360°$$
$$f = \text{frequency [Hz; spins/second]}$$

The imaginary part of the vector corresponds to **instantaneous value** of the sine wawe, while the modulus *(which is constant)* corresponds to the **peak value**, that is positive at 90° and negative at 270°.

The example below is about a voltage of 100 V. The vector performs a rotation of 360° in 24 milliseconds, so the frequency is 41,7 Hz *(f=1/T).*



How to solve AC circuits



All first we have to calculate inductive and capacitive reactances *(X_L , X_C)*:

$$X_L = 2\pi f L = 2\Pi * 50 * 0,6 = 188,5\Omega$$

$$X_{C1} = \frac{1}{2\pi f C_1} = \frac{1}{2\Pi * 50 * 50 * 10^{-6}} = 63,7\Omega$$

$$X_{C2} = \frac{1}{2\pi f C_2} = \frac{1}{2\Pi * 50 * 60 * 10^{-6}} = 53,1\Omega$$

$Z_{R1} = 70$ $\quad$ $Z_L = 188,5J$ $\quad$ $Z_{C1} = -63,7J$

$Z_{C2} = -53,1J$ $\quad$ $Z_{R2} = 80$

$|Z_R| = R$ $\quad$ $|Z_C| = X_C$ $\quad$ $|Z_L| = X_L$

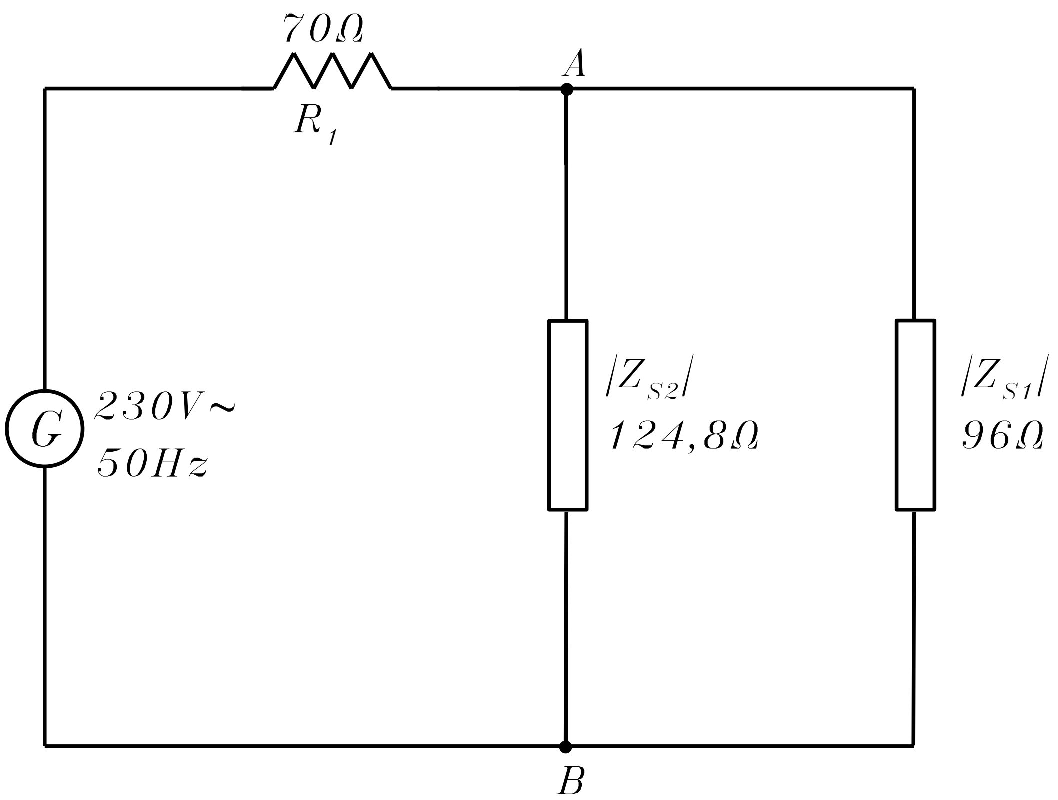
Since we use current and voltage divider we have to calculate the equivalent impedance.

Z_{C2} is connected in series with Z_{R2}, while Z_L is connected in series with Z_{C1}:

$$Z_{S1}=Z_{C2}+Z_{R2}=80-53,1J \qquad |Z_{S1}|=\sqrt{80^2+53,1^2}=96\Omega$$

$$Z_{S2}=Z_L+Z_{C1}=188,5J-63,7J=124,8J$$

$$|Z_{S2}|=124,8\Omega$$



Z_{S1} and Z_{S2} are connected in parallel:

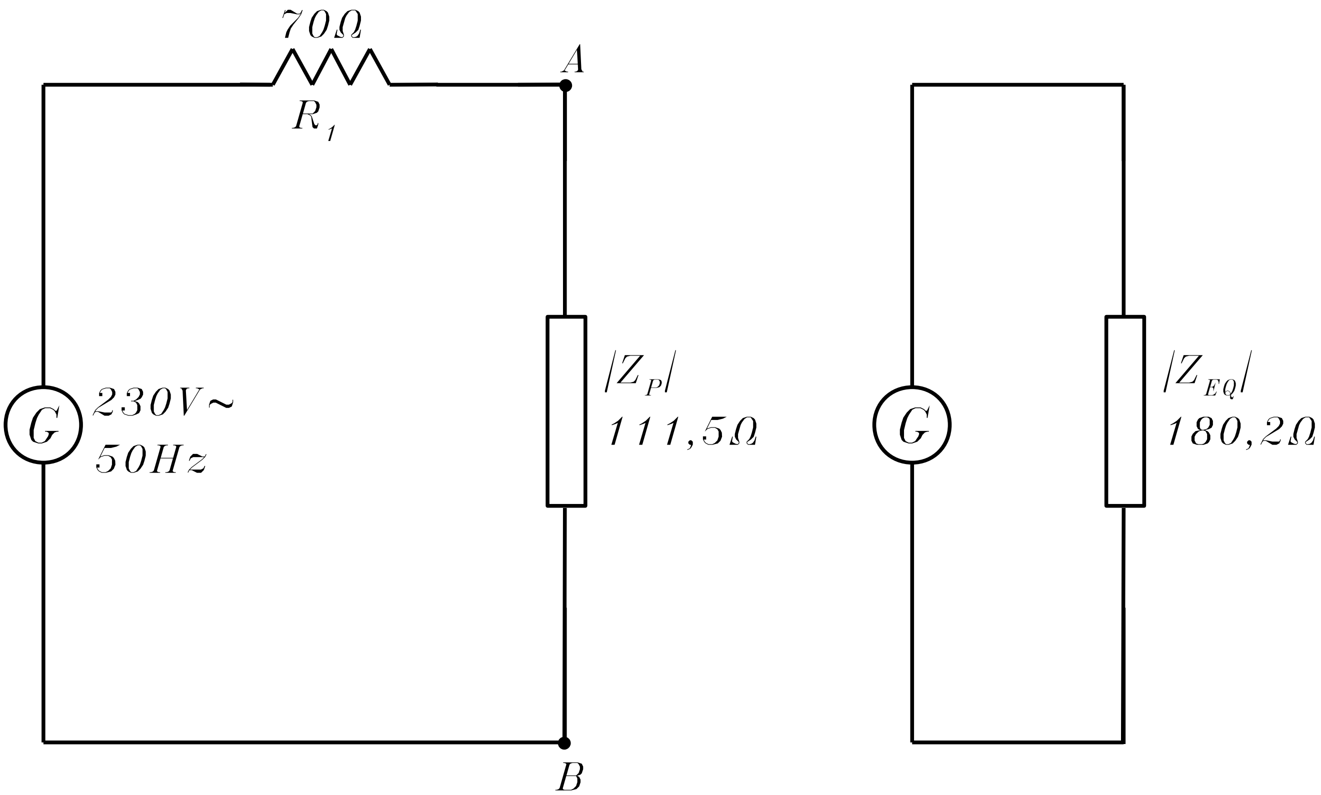
$$Z_P = \frac{Z_{S1} * Z_{S2}}{\overline{Z}_{S1} + \overline{Z}_{S2}} = \frac{(80-53,1J)*124,8J}{80-53,1J+124,8J} = 108+28J$$

$$|Z_P| = \sqrt{108^2+28^2} = 111,5\Omega$$

Z_P and Z_{R1} are connected in series:

$$Z_{EQ} = Z_{R1} + Z_P = 70+108+28J = 178+28J$$

$$|Z_{EQ}| = \sqrt{178^2+28^2} = 180,2\Omega$$



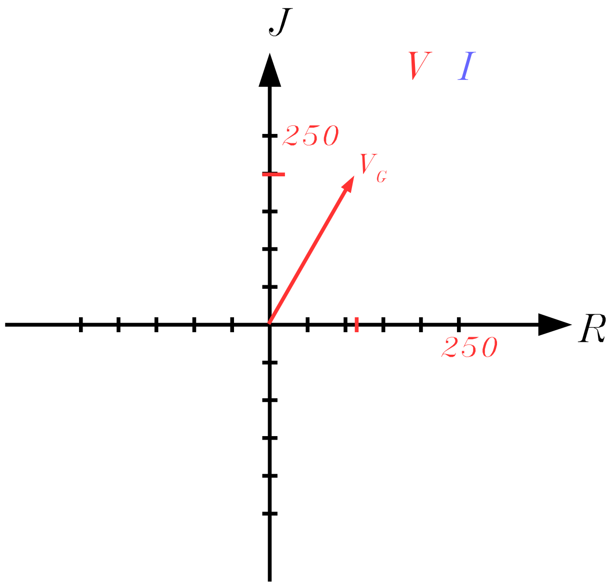
While a vector related to impedance features a constant position, a vector related to voltage or current rotates on the axes intersection. Since we start from the source V_G, we have to chose at will the position that the vector V_G assumes at the instant t_0. By using Ohm's and Kirchhoff's law with complex numbers we can calculate the currents flowing through the circuit at the instant t_0.

$|V_G| = 230V$ *(peak value =±230V)* $\varphi_{VG} = 60°$ $V_G = a + bJ$

$a = |V_G| * \cos\varphi = 230 * \cos 60° = 115$

$b = |V_G| * \sin\varphi = 230 * \sin 60° = 199,2$

$V_G = 115 + 199,2J$ $\boldsymbol{V_{G(t)} = 199,2V}$



At the instant t_0 the voltage V_G features 60° *(chosen at will)* and 199,2V.
Using Ohm's Law:

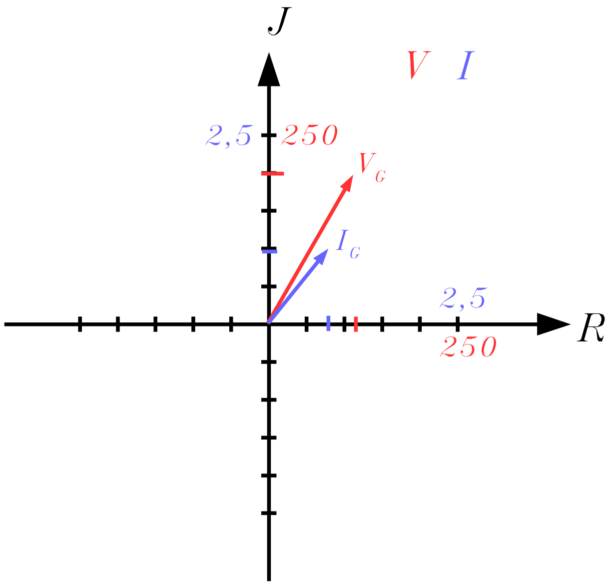
$$I_G = \frac{V_G}{\overline{Z}_{EQ}} = \frac{115 + 199,2J}{178 + 28J} = 0,8 + 0,99J \qquad \boldsymbol{I_{G(t)} = 0,99A}$$

$$|I_G| = \sqrt{0,8^2 + 0,99^2} = 1,27A$$

$$\cos\varphi_{IG} = \frac{0,8}{1,27} = 0,63$$

$$\sin\varphi_{IG} = \frac{0,99}{1,27} = 0,78$$

$$\varphi_{IG} = 51°$$



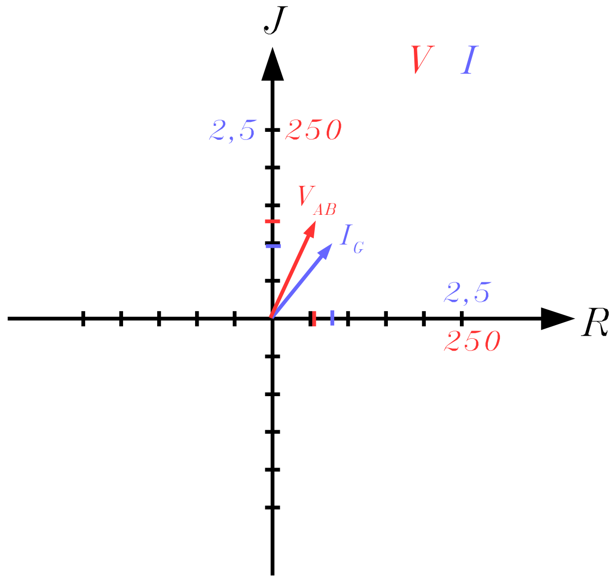
At the instant t_0 the current I_G features 51° and 0,99A
The peak value of the current I_G *(at 90° and 270°)* is ±1,27A

Using voltage divider:

$$V_{AB}=I_G*Z_P=(0,8+0,99J)*(108+28J)=58,7+129,3J$$

$$\boldsymbol{V_{AB(t)}=129,3V}$$

$|V_{AB}|=142V$ $\quad \varphi_{VAB}=65,6^o$

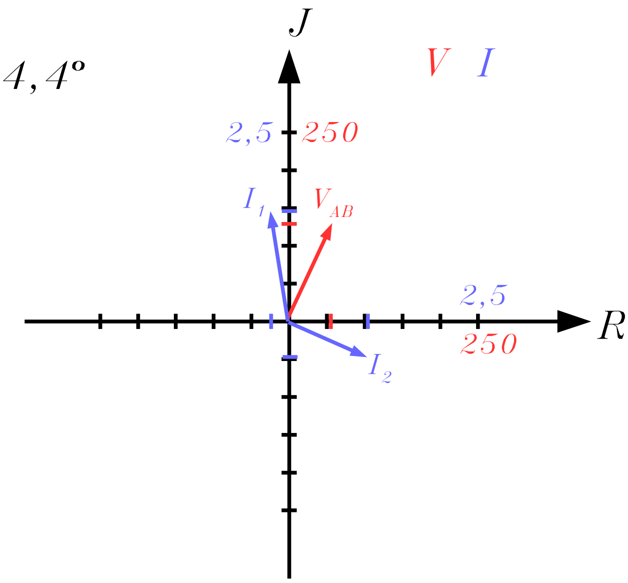


$$I_1=\frac{V_{AB}}{Z_{S1}}=\frac{58,7+129,3J}{80-53,1J}=-0,24+1,46J \qquad \boldsymbol{I_{1(t)}=1,46A}$$

$|I_1|=1,48A$ $\quad \varphi_{I1}=99,2^o$

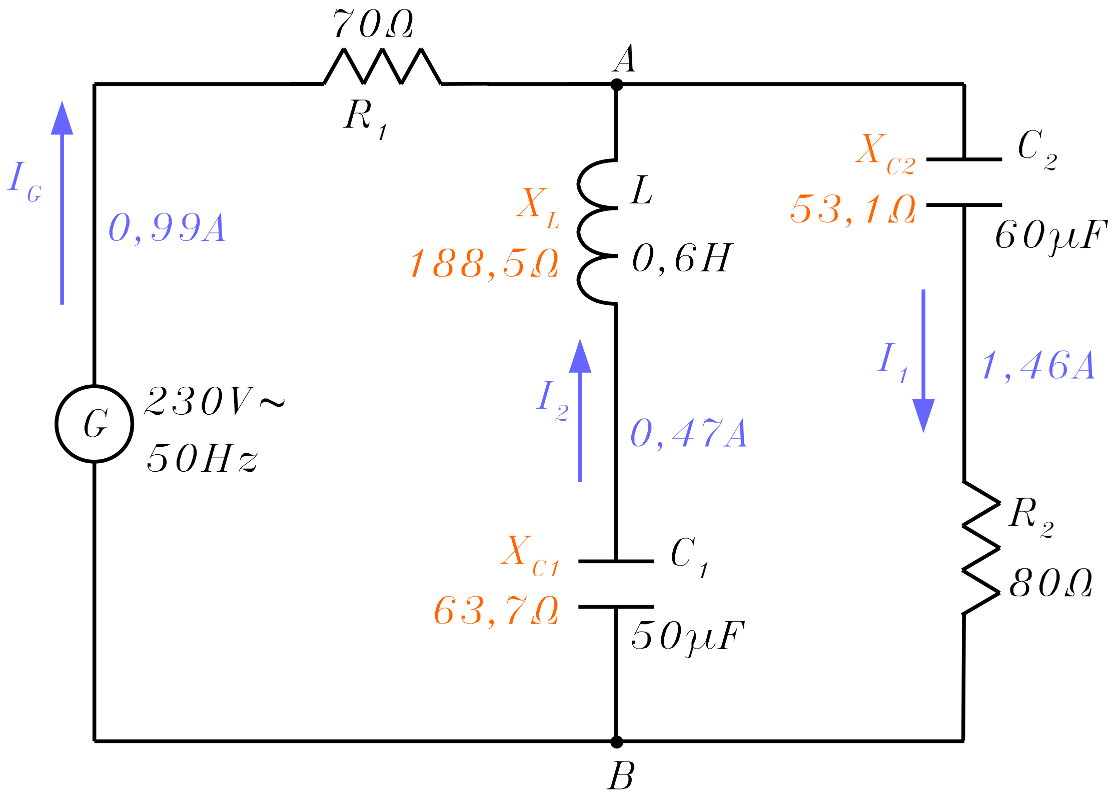
$$I_2=\frac{V_{AB}}{Z_{S2}}=\frac{58,7+129,3J}{124,8J}=1,04-0,47J \qquad \boldsymbol{I_{2(t)}=-0,47A}$$

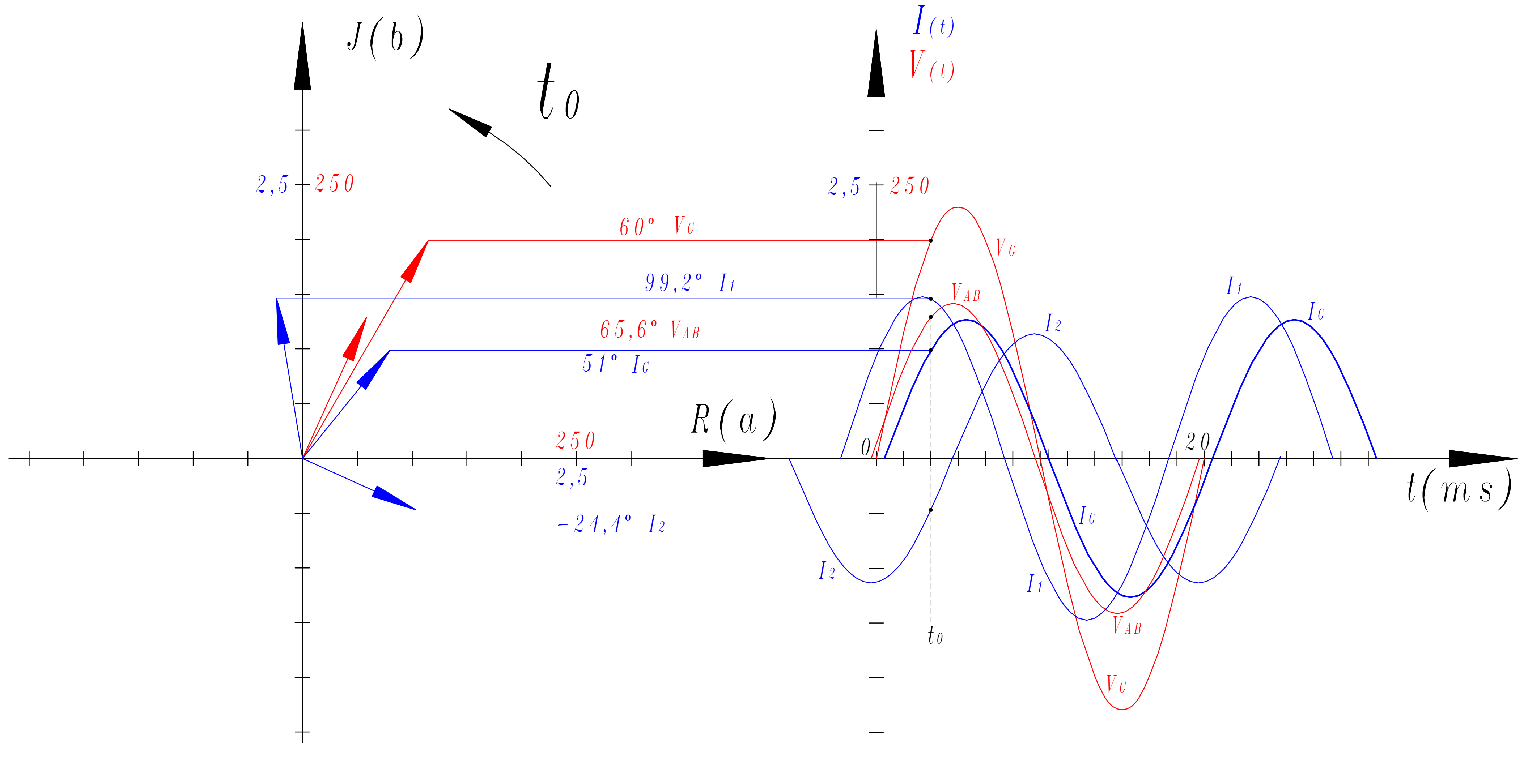
$|I_2|=1,14A$ $\quad \varphi_{I2}=-24,4^o$



The direction of the currents can be chosen at will, but we have to respect Kirchhoff's current law: $0,99A+0,47A=1,46A$

The schematic below shows the currents at the instant t_0:




J(b)
t0
R(a)
2,5
250
60° VG
99,2° I1
65,6° VAB
51° IG
-24,4° I2
I(t)
V(t)
VG
VAB
I1
I2
IG
0
20
t0
t(ms)